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Dynamic Hotelling Monopoly with Product Development[1]

Luca Lambertini

Department of Economics

University of Bologna

Strada Maggiore 45

40125 Bologna, Italy

fax: +39-051-2092664

e-mail: lamberti@spbo.unibo.it

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Abstract

I characterise R&D investment in product innovation of a profit-seeking monopolist versus that of a social planner in a spatial market, under either partial or full market coverage. Under partial coverage, the steady state product design is the outcome of the tradeoff between the incentive to locate as close as possible to the middle of the preference space, and the incentive to save upon R&D costs. The planner does not produce the variety preferred by the average consumer, in situations where the R&D investment is too costly. This result is reinforced under full market coverage, where the planner's incentive to innovate is always weaker than the monopolist's, and the planner produces the average (and median) consumer's preferred variety if and only if the rental price of capital is nil.

Keywords: horizontal differentiation, R&D, steady state, saddle point

JEL Classification: D24, L12, O31

1 Introduction

The analysis of dynamic monopoly dates back to Evans (1924) and Tintner (1937), who analysed the pricing behaviour of a firm subject to a U-shaped variable cost curve.[1] The analysis of intertemporal capital accumulation appeared later on (see Eisner and Strotz, 1963, inter alia).

The existing literature investigates several features of monopoly markets, in particular several forms of discrimination, either through (intertemporal) pricing (see Stokey, 1981; Bulow, 1982; Gul, Sonnenschein and Wilson, 1986) or through product proliferation (see Mussa and Rosen, 1978; Maskin and Riley, 1984; Gabszewicz, Shaked, Sutton and Thisse, 1986; Bonanno, 1987). A dynamic model of durable-good monopoly with capital accumulation in continuous time is in Kamien and Schwartz (1979).

Another dynamic tool which has received a considerable amount of attention is advertising, ever since Vidale and Wolfe (1957) and Nerlove and Arrow (1962).[2] A taxonomy introduced by Sethi (1977) distinguishes between advertising capital models and sales-advertising response models. The first category considers advertising as an investment in a stock of goodwill, à la Nerlove-Arrow. The second category gathers models where there exists a direct relationship between the rate of change in sales and advertising, à la Vidale-Wolfe.

[1] See Chiang (1992) for a recent exposition of the original model by Evans, as well as later developments.

[2] For exhaustive surveys, see Sethi (1977); Jørgensen (1982); Feichtinger and Jørgensen (1983); Erickson (1991); Feichtinger, Hartl and Sethi (1994). For duopoly models with dynamic pricing and advertising, see in particular Leitmann and Schmitendorf (1978), and Feichtinger (1983).

A wide attention has been devoted to the issue of optimal product design in static models of product di¤erentiation, be that either horizontal (Bonanno, 1987) or vertical (Spence, 1975; Mussa and Rosen, 1978; Maskin and Riley, 1984; Gabszewicz, Shaked, Sutton and Thisse, 1986). According to these contributions, a pro…t-seeking monopolist designs the product to …t the taste of the marginal consumer, while a benevolent planner aiming at the maximization of social welfare takes into account the taste of the average consumer. Therefore, whenever the marginal willingness to pay of the average consumer is higher (lower) than the marginal consumer's, we observe a downward (upward) distortion in the equilibrium design of the product. A situation where this does not happen is the horizontal di¤erentiation model à la Hotelling (1929) investigated by Bonanno (1987), where all consumers have the same gross surplus. Consequently, the …rm locates in the middle of the product space irrespective of her objective function.

In this paper, I propose a monopoly model where the …rm locates the product in a spatial market representing the space of consumer preferences, as in Hotelling (1929). The location of the product in the space of consumer preferences depends upon the R&D investment carried out by the …rm over time, according to a technology characterised by decreasing returns.

The analysis is carried out under the alternative assumptions of partial and full market coverage. In the …rst case, the steady state product design is the outcome of the tradeo¤ between the incentive to locate as close as possible to the middle of the preference space, and the incentive to save upon R&D costs. The planner does not produce the variety preferred by the average consumer, in situations where the R&D investment is too costly.

This conclusion emerges even more clearly from the full coverage framework, where the planner's incentive to invest in product innovation is always weaker than the monopolist's, and the planner produces the average (and median) consumer's preferred variety if and only if the rental price of capital is nil.

Hence, in general, there are situations where the R&D effort needed to meet the tastes of the average consumer is socially too expensive while, on the contrary, the monopolist finds it convenient to locate her product in the middle of the preference space in order to maximise the extraction of surplus from consumers. An ancillary but relevant corollary is that, whenever the monopolist's variety is closer to the average consumer taste than the planner's, the extent of market coverage at the steady state is larger under monopoly than under social planning. Therefore, while it is certainly true that welfare is higher under planning than under monopoly, this happens because of a smaller R&D effort in product innovation rather than a larger production and/or a more efficient location of the product in the admissible spectrum of varieties.

The remainder of the paper is structured as follows. The setup is laid out in section 2. The behaviour of the planner and the monopolist under partial market coverage is illustrated in section 3. Section 4 contains a comparative evaluation of their performances in steady state. The full market coverage setting is investigated in section 5, and the comparative assessment of social planning and monopoly is in section 6. Section 7 contains concluding remarks.

2 The model

The setup shares its basic features with Bonanno (1987). I consider a market for horizontally differentiated products where consumers are uniformly distributed with unit density along the unit interval $[0; 1]$. Let the market exist over $t \in [0; \infty)$. The market is served by a firm selling a single good, located at $x(t) \in [1/2; 1]$. This assumption is justified by the symmetry of the model around 1/2. The initial condition is $x(0) = 1$; and the firm may modify its location over time according to the following dynamics of the state variable:[3]

$$\frac{\partial x(t)}{\partial t} = -\frac{k(t)}{1 + k(t)} \cdot x(t) \; ; \tag{1}$$

where $k(t)$ is the R&D effort carried out by the firm at time t. Observe that technology (1) exhibits decreasing returns to scale.

The generic consumer located at $a(t) \in [0; 1]$ buys one unit of the good, if net surplus from purchase is non-negative:

$$U(t) = s - p(t) - [x(t) - a(t)]^2 \geq 0; \qquad i = 1; 2; \tag{2}$$

where $p(t)$ is the firm's mill price, and s is gross consumer surplus, that is, the reservation price that a generic consumer is willing to pay for the good. Therefore, s can be considered as a preference parameter which, together with the disutility of transportation, yields a measure of consumers' taste for the good.

Two alternative situations may arise:

[3] A stochastic R&D race for product innovation in a Hotelling duopoly with quadratic disutility of transportation is investigated by Harter (1993).

[1] **Partial market coverage (pmc)**: The mill price is such that marginal consumer located at $m(t) \in (0; x(t))$ enjoys zero surplus, that is,

$$p(t) = s - [x(t) - m(t)]^2 \tag{3}$$

This gives rise to a market demand equal to:

$$y(t) = 1 - m(t) . \tag{4}$$

[2] **Full market coverage (fmc)**: The reservation price s is sufficiently high to allow for the firm to serve all consumers, so that demand is $y(t) = 1$. In such a case, for all $x(t) \in \left[\frac{1}{2}; 1\right]$, the monopoly price is $p(t) = s - [x(t)]^2$.

I assume that the firm operates at constant marginal production cost, and, for the sake of simplicity, I normalise it to zero.[4] Therefore, instantaneous profits are:

$$\pi(t) = p(t)y(t) = \begin{cases} \left[s - (x(t) - m(t))^2\right][1 - m(t)] - \rho k(t) \text{ under pmc} \\ s - [x(t)]^2 - \rho k(t) \text{ under fmc} \end{cases} \tag{5}$$

where ρ is the rental price of capital, assumed to be equal to the discount rate. Instantaneous consumer surplus is:

$$\begin{aligned} CS(t) &= \int_{m(t)}^{1} \left[s - p(t) - (x(t) - a(t))^2\right] da(t) = \\ &= \frac{\left[1 - (m(t))^2\right][3x(t)m(t) - 2m(t) - 1]}{3} \end{aligned} \tag{6}$$

---

[4] The properties of capital accumulation for both advertising and production in a similar monopoly setting are investigated in Lambertini (2000).

under partial market coverage. Instantaneous social welfare under partial coverage amounts to

$$SW(t) = \pi(t) + CS(t)\,. \tag{7}$$

Under full market coverage, the (inverse) measure of social welfare is given by the integral of transportation costs over the population of consumers:

$$TC(t) = \int_0^1 [x(t) - a(t)]^2\, da(t) = [x(t)]^2 - x(t) + \frac{1}{3}\,. \tag{8}$$

Notice that, on the basis of (8), the social planner can set any price $p(t) \in \left[0; s - (x(t))^2\right]$.

3 Product development under partial market coverage

3.1 Socially optimal R&D

In scenario 1, the objective of a benevolent social planner is

$$\max_{a(t)} \int_0^\infty e^{-\rho t} SW(t)dt = \int_0^\infty e^{-\rho t} \left\{ \frac{\left[1 - (m(t))^2\right][3x(t)m(t) - 2m(t) - 1]}{3} + \right. \\ \left. + \left[s - (x(t) - m(t))^2\right][1 - m(t)] - \rho k(t) \right\} dt \tag{9}$$

$$s.t. \quad \frac{\partial x(t)}{\partial t} = -\frac{k(t)}{1 + k(t)} \cdot x(t) \tag{10}$$

where ρ denotes time discounting. In choosing the optimal location of the marginal consumer(s) at any t; the planner indeed maximises discounted social welfare w.r.t. output (or, alternatively, price). The corresponding Hamiltonian function is:

$$H(t) = e^{-\rho t} \cdot \left\{ \frac{\left[1 - (m(t))^2\right][3x(t)m(t) - 2m(t) - 1]}{3} + \right. \tag{11}$$

$$+\left[s - (x(t) - m(t))^2\right][1 - m(t)] - \rho k(t) - \lambda(t)\frac{k(t)}{1+k(t)} \cdot x(t) \Big\} ;$$

where $\lambda(t) = \beta(t)e^{\rho t}$; $\beta(t)$ being the co-state variable associated to $k(t)$: The necessary and sufficient conditions for a path to be optimal are:

$$\frac{\partial H(t)}{\partial m(t)} = [m(t)]^2 + 2m(t)x(t) + [x(t)]^2\, 2 - s = 0 ; \tag{12}$$

$$\frac{\partial H(t)}{\partial k(t)} = -\rho - \frac{\lambda(t)}{[1+k(t)]^2} \cdot x(t) = 0 ; \tag{13}$$

$$\begin{aligned} -\frac{\partial H(t)}{\partial x(t)} &= \frac{\partial \beta(t)}{\partial t} \Rightarrow \\ \frac{\partial \lambda(t)}{\partial t} &= \left(\rho + \frac{k(t)}{1+k(t)}\right)\lambda(t) - \frac{[1-m(t)][3+3m(t)-6x(t)]}{3} ; \end{aligned} \tag{14}$$

$$\lim_{t \to \infty} \beta(t) \cdot x(t) = 0 : \tag{15}$$

From (12), I obtain[5]

$$m(t) = x(t) - \sqrt{s} \tag{16}$$

with $m(t)$ smaller than $x(t)$ for all positive s; and $m(t) > 0$ for all $x(t) \in (1/2; 1]$ iff $s < 1/4$:

From (13), I obtain

$$\lambda(t) = -\frac{\rho[1+k(t)]^2}{x(t)} \tag{17}$$

and

$$k(t) = -1 + \frac{1}{\sqrt{\rho}}\sqrt{-\lambda(t)x(t)} \tag{18}$$

which allows me to write the following dynamics of capital accumulation:

$$\frac{\partial k(t)}{\partial t} = \frac{1}{2\sqrt{-\rho\lambda(t)x(t)}}\left\{-\frac{\partial \lambda(t)}{\partial t}x(t) - \frac{\partial x(t)}{\partial t}\lambda(t)\right\} \tag{19}$$

[5] Recall that $m(t) < x(t) \in (1/2; 1]$; so that the other solution to (12) can be excluded.

which in turn entails:

$$\frac{\partial k(t)}{\partial t} \propto -\frac{\partial \lambda(t)}{\partial t}x(t) - \frac{\partial x(t)}{\partial t}\lambda(t) \, . \tag{20}$$

Using (14) and (17), condition (13) rewrites as follows:

$$\frac{\partial k(t)}{\partial t} \propto \rho^2 [1 + k(t)]^2 + x(t) [1 - m(t)] [1 + m(t) - 2x(t)] \, . \tag{21}$$

The roots of the expression on the r.h.s. of (21) are:

$$k(t) = -1 \pm \frac{\sqrt{x(t) [1 - m(t)] [2x(t) - 1 + m(t)]}}{\rho} \tag{22}$$

and using (16):

$$k(t) = -1 \pm \frac{\sqrt{x(t) \{s - 1 + x(t) [2 - x(t)]\}}}{\rho} \, . \tag{23}$$

First notice that obviously the negative root can be excluded. Secondly, observe that

$$k(t) \in \mathbb{R}^+ \text{ for all } x(t) \in \left[1 - \sqrt{s}\,; 1 + \sqrt{s}\right] \, . \tag{24}$$

If condition (24) does not hold, then $\partial k(t)/\partial t$ is always positive. The foregoing discussion produces:

Lemma 1 For all $x(t) \in [1 - \sqrt{s}\,; 1 + \sqrt{s}]$, the socially optimal R&D investment in steady state is:

$$k^{ss}_{SP} = -1 + \frac{\sqrt{x(t) \{s - 1 + x(t) [2 - x(t)]\}}}{\rho} \, , \tag{25}$$

with the intuitive properties:

$$\frac{\partial k^{ss}_{SP}}{\partial s} > 0 \, ; \; \frac{\partial k^{ss}_{SP}}{\partial \rho} < 0 \, .$$

Superscript ss stands for steady state while subscript SP stands for social planning. From (22), I also obtain that $k_{SP}^{ss} = 0$ at[6]

$$x(t) = \frac{1 - [m(t)]^2 + \sqrt{[1 - m(t)]\left\{1 + 8\rho^2 - m(t)\left[(m(t))^2 + m(t) - 1\right]\right\}}}{4\,[1 - m(t)]} \tag{26}$$

while, from (25), $k_{SP}^{ss} = 0$ at[7]

$$x_{SP}^{ss} = \frac{2}{3}\left(1 + \frac{1 + 3s + \sqrt[3]{2\Omega}}{\sqrt[3]{4\Omega}}\right) \tag{27}$$

where

$$\Omega \equiv 18s - 2 - 27\rho^2 + 3\sqrt{3\left[\rho^2\,(4 + 27\rho^2 - 36s) - 4s\,(1 - s)^2\right]}. \tag{28}$$

Some tedious but easy algebra is necessary to verify the following results:

- At $x(0) = 1$, $k_{SP}^{ss} = \dfrac{\sqrt{s} - \rho}{\rho} > 0$ for all $s > \rho^2$.
- $k_{SP}^{ss} = 0$ at $x_{SP}^{ss} = \dfrac{2}{3}\left(1 + \dfrac{1 + 3s + \sqrt[3]{2\Omega}}{\sqrt[3]{4\Omega}}\right) \in \left(\dfrac{1}{2}, 1\right]$ for all $s \in \left[\rho^2, \dfrac{8\rho^2 + 1}{4}\right)$.
- $k_{SP}^{ss} = 0$ at $x_{SP}^{ss} = \dfrac{1}{2}$, for $s = \dfrac{8\rho^2 + 1}{4}$.
- $k_{SP}^{ss} = -1 + \dfrac{1}{2\rho}\sqrt{\dfrac{4s - 1}{2}}$ at $x(t) = \dfrac{1}{2}$, for all $s > \dfrac{8\rho^2 + 1}{4}$.

The above can be summarised by:

[6] The smaller root can be disregarded as it is always negative.

[7] The other two roots can be disregarded as they are both complex.

Proposition 1 *For all $s \in \left[\rho^2; \frac{8\rho^2+1}{4}\right)$; the social planning problem admits a unique steady state equilibrium where $k_{SP}^{ss} = 0$ at $x_{SP}^{ss} \in \left(\frac{1}{2}; 1\right]$.*

Examine now the extent of market coverage in steady state. This is (inversely) measured by $m_{SP}^{ss} = x_{SP}^{ss} - \sqrt{s}$. Now recall that $m(t) > 0$ for all $s \in \left(0; \frac{1}{4}\right)$; and observe that $\frac{8\rho^2+1}{4} > \frac{1}{4}$ for all $\rho > 0$. This yields the following Corollary to Proposition 1:

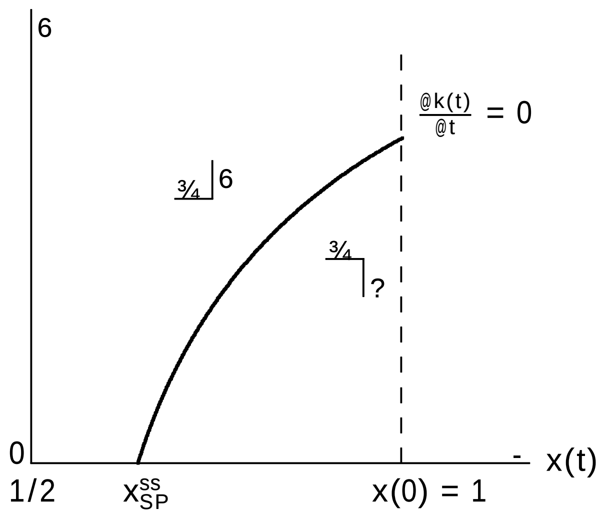
Corollary 1 *For all $\rho > 0$ and $s \in \left[\rho^2; \frac{8\rho^2+1}{4}\right)$; $k_{SP}^{ss} = 0$ at $x_{SP}^{ss} \in \left(\frac{1}{2}; 1\right]$; and $m_{SP}^{ss} \in (0; x_{SP}^{ss})$. Partial market coverage obtains.*

For $\rho = 0$ and $s = \frac{1}{4}$; in steady state $\left\{k_{SP}^{ss} = 0; x_{SP}^{ss} = \frac{1}{2}; m_{SP}^{ss} = 0\right\}$. Full market coverage obtains as an internal solution.

For all $\rho > 0$ and $s > \frac{8\rho^2+1}{4}$; in steady state $k_{SP}^{ss} > 0$; with $x = \frac{1}{2}$ and $m = 0$. Full market coverage obtains as a corner solution.

The R&D behaviour and the resulting optimal location choice of the planner are illustrated in figure 1, describing the case $s \in \left[\rho^2; \frac{8\rho^2+1}{4}\right)$. The dynamics of $x(t)$ and $k(t)$ are summarised by horizontal and vertical arrows, respectively. This also suffices to show that, whenever $x_{SP}^{ss} \in \left(\frac{1}{2}; 1\right]$; then it is a saddle.

Figure 1 : Dynamics in the space $(x(t); k(t))$



3.2 Optimal R&D in a profit-seeking monopoly

The objective of the monopolist is

$$\max_{a(t)} \int_0^\infty e^{-\rho t}\pi(t)dt = \int_0^\infty e^{-\rho t}\left\{\left[s - (x(t) - m(t))^2\right][1 - m(t)] - \rho k(t)\right\} dt \tag{29}$$

$$s.t.: \quad \frac{\partial k(t)}{\partial t} = f(k(t)) - q(t) - \delta k(t) \tag{30}$$

where ρ denotes the same time discounting as for the planner. The corresponding Hamiltonian function is:

$$H(t) = e^{-\rho t}\cdot\left\{\left[s - (x(t) - m(t))^2\right][1 - m(t)] - \rho k(t) - \lambda(t)\frac{k(t)}{1 + k(t)}\cdot x(t)\right\}; \tag{31}$$

where, again, $\lambda(t) = \beta(t)e^{\rho t}$; and $\beta(t)$ is the co-state variable associated to $k(t)$:

The solution to the monopolist's problem is largely analogous to that of the planner as illustrated in section 3.1. The first order conditions are:

$$\frac{\partial H(t)}{\partial m(t)} = 3[m(t)]^2 - 2[1+2x(t)]m(t) + [x(t)]^2 - 2x(t) - s = 0; \qquad (32)$$

$$\frac{\partial H(t)}{\partial k(t)} = -\rho - \frac{\lambda(t)}{[1+k(t)]^2} \cdot x(t); \qquad (33)$$

$$-\frac{\partial H(t)}{\partial x(t)} = \frac{\partial \beta(t)}{\partial t} \Rightarrow \qquad (34)$$

$$\frac{\partial \lambda(t)}{\partial t} = \left[\rho + \frac{k(t)}{1+k(t)}\right]\lambda(t) + 2\{x(t) - m(t)[1+m(t)+x(t)]\};$$

$$\lim_{t\to\infty} \beta(t) \cdot x(t) = 0: \qquad (35)$$

Condition (32) yields

$$m(t) = \frac{1+2x(t) - \sqrt{[1-x(t)]^2+3s}}{3}; \qquad (36)$$

while from condition (33) I obtain qualitatively the same dynamics for the R&D effort as observed under planning:

$$\frac{\partial k(t)}{\partial t} \propto -\frac{\partial \lambda(t)}{\partial t}x(t) - \frac{\partial x(t)}{\partial t}\lambda(t); \qquad (37)$$

which rewrites as follows:

$$\frac{\partial k(t)}{\partial t} \propto \rho^2[1+k(t)]^2 - x(t)[x(t)-m(t)][1-m(t)]: \qquad (38)$$

The roots of the expression on the r.h.s. of (38) are:

$$k(t) = -1 \pm \frac{\sqrt{2x(t)[x(t)-m(t)][1-m(t)]}}{\rho}: \qquad (39)$$

The negative root can obviously be disregarded, so that the optimal R&D effort in steady state is:

$$k_M^{ss} = -1 + \frac{\sqrt{2x(t)\,[x(t) - m(t)]\,[1 - m(t)]}}{\rho} \, ; \tag{40}$$

where subscript M stands for monopoly.[8] Moreover,

$$k_M^{ss} = 0 \quad , \quad x(t) = \frac{m(t) \pm \sqrt{[m(t)]^2 + \dfrac{2\rho^2}{[1 - m(t)]}}}{2} \tag{41}$$

which obviously entails:

$$x_M^{ss} = \frac{m(t) + \sqrt{[m(t)]^2 + \dfrac{2\rho^2}{[1 - m(t)]}}}{2} \, . \tag{42}$$

By using (36) and (40), the following properties can be shown to hold:

- At $x(0) = 1$; $k_M^{ss} = -1 + \frac{1}{\rho}\sqrt{\frac{2}{3}s} > 0$ for all $s > \frac{3}{2}\rho^2$.
- $k_M^{ss} = 0$ at $x_M^{ss} = \dfrac{m(t) + \sqrt{[m(t)]^2 + \dfrac{2\rho^2}{[1 - m(t)]}}}{2} \in \left(\frac{1}{2}; 1\right)$ for all $s \in \left(\frac{3}{2}\rho^2 ; \min\left\{\frac{5}{4}; \frac{24\rho^2 + 1 - \sqrt{16\rho^2 + 1}}{8}\right\}\right)$.
- $k_M^{ss} = 0$ at $x_M^{ss} = \frac{1}{2}$, for $s = \min\left\{\frac{5}{4}; \frac{24\rho^2 + 1 - \sqrt{16\rho^2 + 1}}{8}\right\}$.
- $k_M^{ss} = -1 + \frac{1}{6\rho}\sqrt{12s - 1 + \sqrt{12s + 1}}$ at $x(t) = \frac{1}{2}$, for all $s > \min\left\{\frac{5}{4}; \frac{24\rho^2 + 1 - \sqrt{16\rho^2 + 1}}{8}\right\}$.

[8] The optimal capital level under monopoly, k_M^{ss}, can be rewritten as a function of $\{x(t), s, \rho\}$ only. However, this expression is cumbersome and therefore it is omitted.

The above can be summarised by:

Proposition 2 For all $s \in \left[\frac{3}{2}\rho^2, \min\left\{\frac{5}{4}, \frac{24\rho^2+1-\sqrt{16\rho^2+1}}{8}\right\}\right)$, the monopoly problem admits a unique steady state equilibrium where $k_M^{ss}=0$ at $x_M^{ss} \in \left(\frac{1}{2}, 1\right]$.

In order to characterise the extent of market coverage in steady state, observe that, from (36), $m(t) > 0$ for all $s \in \left(0, \frac{5}{4}\right)$, with

$$\begin{aligned} \frac{24\rho^2+1-\sqrt{16\rho^2+1}}{8} &< \frac{5}{4} \text{ for all } \rho \in \left[0, \frac{\sqrt{2}}{2}\right), \\ \frac{24\rho^2+1-\sqrt{16\rho^2+1}}{8} &\geq \frac{5}{4} \text{ for all } \rho \geq \frac{\sqrt{2}}{2}. \end{aligned} \tag{43}$$

Therefore, we have the following Corollary to Proposition 2:

Corollary 2 For all $\rho \in \left[0, \frac{\sqrt{2}}{2}\right)$ and $s \in \left[\frac{3}{2}\rho^2, \frac{24\rho^2+1-\sqrt{16\rho^2+1}}{8}\right)$, $k_M^{ss}=0$ at $x_M^{ss} \in \left(\frac{1}{2}, 1\right]$, and $m_M^{ss} \in (0, x_M^{ss})$: Partial market coverage obtains.

For all $\rho \in \left[0, \frac{\sqrt{2}}{2}\right)$ and $s \in \left[\frac{24\rho^2+1-\sqrt{16\rho^2+1}}{8}, \frac{5}{4}\right)$, $k_M^{ss} > 0$ at $x_M^{ss} = \frac{1}{2}$, and $m_M^{ss} \in (0, x_M^{ss})$: Partial market coverage obtains.

For all $\rho \in \left[0, \frac{\sqrt{2}}{2}\right)$ and $s \geq \frac{5}{4}$, $k_M^{ss} > 0$ at $x_M^{ss} = \frac{1}{2}$, $m_M^{ss} = 0$: Full market coverage obtains.

For $\rho = \frac{\sqrt{2}}{2}$ and $s = \frac{5}{4}$, in steady state $\left\{k_{SP}^{ss} = 0, x_{SP}^{ss} = \frac{1}{2}, m_{SP}^{ss} = 0\right\}$: Full market coverage obtains as an internal solution.

For all $\rho > \frac{\sqrt{2}}{2}$ and $s > \frac{5}{4}$, in steady state $k_{SP}^{ss} > 0$, with $x = \frac{1}{2}$ and $m = 0$: Full market coverage obtains as a corner solution.

The dynamic analysis in the space $\{x(t); k(t)\}$ shows that, whenever $x_M^{ss} \in \left(\frac{1}{2}; 1\right)$; then it is a saddle. The phase diagram is qualitatively equivalent to figure 1.

4 Planning vs monopoly under partial market coverage

The above analysis can be summarised so as to give a comparative evaluation of the planner's and the monopolist's behaviour, as follows. First of all, the following result stems from the straightforward comparison between k_M^{ss} and k_{SP}^{ss} at $x(0) = 1$:

Lemma 2 *At $t = 0$; the optimal instantaneous investment is larger under social planning than under monopoly, for all admissible values of $\{s; \rho\}$:*

That is, the social incentive towards investment in product development is initially larger than the monopolist's. Moreover:

I. For all $s \in [0; \rho^2]$; neither the planner nor the monopolist invest in product innovation, and $x_{SP}(t) = x_M(t) = 1$ forever.

II. For all $s \in \left(\rho^2; \frac{3}{2}\rho^2\right)$; the planner carries out R&D for product innovation, thereby obtaining $x_{SP}^{ss} \in \left(\frac{1}{2}; 1\right)$; while the monopolist does not invest.

III. For all

$$s \in \left(\frac{3}{2}\rho^2; \min\left\{\frac{8\rho^2+1}{4}; \frac{24\rho^2+1-\sqrt{16\rho^2+1}}{8}; \frac{5}{4}\right\}\right];$$

we have the following cases:

A] If $\rho \in \left[0;\ \frac{\sqrt{2}}{2}\right)$; then $\min\left\{\frac{8\rho^2+1}{4};\ \frac{24\rho^2+1-\sqrt{16\rho^2+1}}{8};\ \frac{5}{4}\right\} = \frac{24\rho^2+1-\sqrt{16\rho^2+1}}{8}$ and $x_i^{ss} \in \left(\frac{1}{2};\ 1\right]$; $m_i^{ss} \in (0;\ x_i^{ss})$; $i = M; SP$: Partial market coverage obtains under both regimes.

Moreover, if $\rho \in [0\ ;\ 0.3067)$; then

$$\frac{1}{2} < x_M^{ss} < x_{SP}^{ss} < 1 \text{ for all } s \in \left(\frac{3}{2}\rho^2;\ \frac{24\rho^2+1-\sqrt{16\rho^2+1}}{8}\right] : \quad (44)$$

That is, there exists a subset of parameters where the monopolist locates closer to the middle of linear city than the planner does.

B] If $\rho \geq \frac{\sqrt{2}}{2}$; then $\min\left\{\frac{8\rho^2+1}{4};\ \frac{24\rho^2+1-\sqrt{16\rho^2+1}}{8};\ \frac{5}{4}\right\} = \frac{5}{4}$ and

$$\frac{1}{2} < x_{SP}^{ss} < x_M^{ss} < 1 \text{ for all } s \in \left(\frac{3}{2}\rho^2;\ \frac{5}{4}\right] \text{ and all } \rho \geq \frac{\sqrt{2}}{2} : \quad (45)$$

That is, in this parameter range, the planner works out a product design that fits the tastes of the median (and average) consumer better than the product supplied by the monopolist.

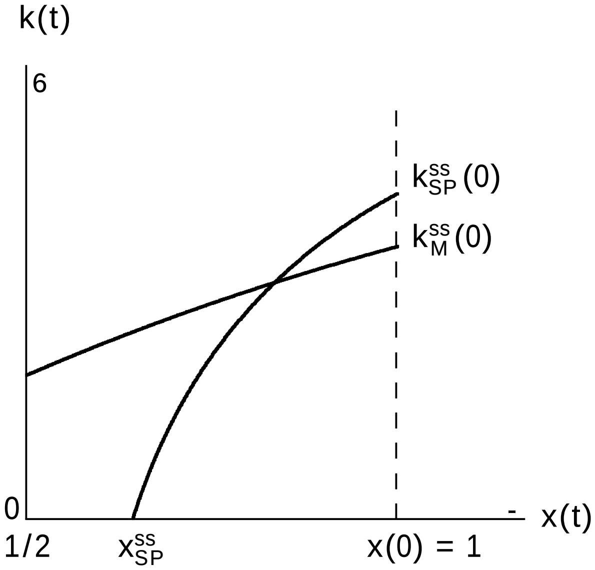
IV. If

$$s \in \left[\min\left\{\frac{8\rho^2+1}{4};\ \frac{24\rho^2+1-\sqrt{16\rho^2+1}}{8};\ \frac{5}{4}\right\};\ \max\left\{\frac{8\rho^2+1}{4};\ \frac{24\rho^2+1-\sqrt{16\rho^2+1}}{8};\ \frac{5}{4}\right\}\right);$$

the following cases arise:

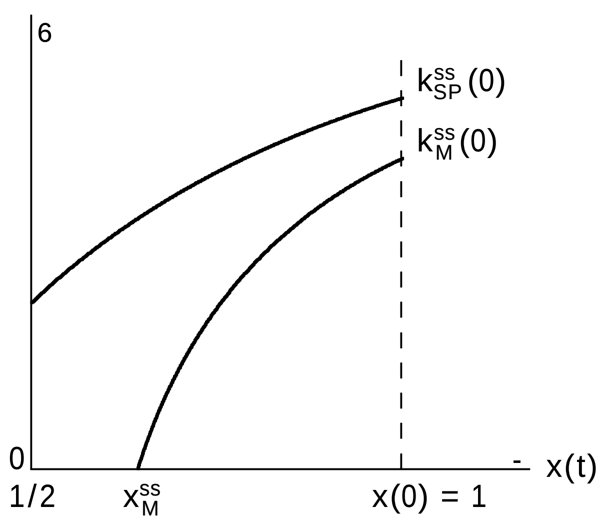
A] If $\rho \in \left(0, \frac{\sqrt{2}}{2}\right)$, then $s \in \left(\frac{24\rho^2 + 1 - \sqrt{16\rho^2 + 1}}{8}, \frac{5}{4}\right)$ and $\left\{x_M^{ss} = \frac{1}{2}; k_M^{ss} > 0\right\}$; $\left\{x_{SP}^{ss} \in \left(\frac{1}{2}, 1\right]; k_{SP}^{ss} = 0\right\}$: This situation is described in figure 2.

Figure 2 : Comparative dynamics in the space $(x(t), k(t))$, $\rho \in \left(0, \frac{\sqrt{2}}{2}\right)$:



B] If $\rho \geq \frac{\sqrt{2}}{2}$, then $s \in \left(\frac{5}{4}, \frac{24\rho^2 + 1 - \sqrt{16\rho^2 + 1}}{8}\right)$ and $\left\{x_M^{ss} \in \left(\frac{1}{2}, 1\right]; k_M^{ss} = 0\right\}$; $\left\{x_{SP}^{ss} = \frac{1}{2}; k_{SP}^{ss} > 0\right\}$: This situation is described in figure 3.

Figure 3 : Comparative dynamics in the space $(x(t); k(t))$, $\rho \geq \frac{\sqrt{2}}{2}$:



V. If $s \geq \max\left\{ \frac{8\rho^2+1}{4}; \frac{24\rho^2+1-\sqrt{16\rho^2+1}}{8}; \frac{5}{4} \right\}$; then a corner solution obtains in both regimes, with $\left\{ x_i^{ss} = \frac{1}{2}; k_{SP}^{ss} \geq k_M^{ss} > 0 \right\}$; $i = M; SP$:

As a matter of curiosity, it is easy to verify that, when $\left\{ s = \frac{5}{4}; \rho = \frac{\sqrt{2}}{2} \right\}$; the planning optimum and the monopoly optimum coincide at the saddle point $\left\{ k_i^{ss} = 0; x_i^{ss} = \frac{1}{2} \right\}$:

The above discussion can be summarised by saying that neither the planner nor the profit-seeking monopolist does necessarily aim at developing the product variety preferred by the average (and median) consumer located at 1/2. In general, they both reach 1/2 for sufficiently high levels of the gross surplus s (case V). Otherwise, the steady state location of the product depends upon the tradeoff between the incentive to locate as close as possible to the middle of the preference space, so as to (i) extract as much surplus as possible, in the monopoly case, and (ii) minimise total transportation disutility, under social planning; and the incentive to save on R&D costs. In particular, the conventional wisdom associated with static models (Spence, 1975; Bonanno, 1987), maintaining that the benevolent planner cares about the average consumer, does not carry over to the present dynamic analysis. Indeed, the planner may not produce the variety preferred by the average consumer, as long as market coverage is sufficiently large. That is, there are situations where the R&D investment needed to reach 1/2 is socially too costly. This is the case if

$$\rho \in \left[0\,;\,\frac{\sqrt{2}}{2}\right) \quad \text{and}$$

$$\min\left\{\frac{8\rho^2+1}{4}\,;\,\frac{24\rho^2+1-\sqrt{16\rho^2+1}}{8}\,;\,\frac{5}{4}\right\} = \frac{24\rho^2+1-\sqrt{16\rho^2+1}}{8}\,.$$

On the contrary, under the same conditions, the monopolist finds it convenient to locate at 1/2 so as to maximise the extraction of surplus from consumers. This also implies the following corollary:

Corollary 3 Whenever $x_M^{ss} < x_{SP}^{ss}$; the extent of market coverage at the steady state is larger under monopoly than under social planning.

Nevertheless, it is obviously true that welfare is higher under planning than under monopoly, because of a smaller R&D effort in product innovation.

5 Product development under full market coverage

5.1 Socially optimal R&D

Under full market coverage, the benevolent planner aims at minimising the integral of transportation costs net of R&D costs, under the kinematics of location as in (10). Accordingly, the relevant Hamiltonian is:

$$H(t) = e^{-\rho t} \cdot \left\{ x(t) - [x(t)]^2 - \frac{1}{3} - \rho k(t) - \lambda(t) \frac{k(t)}{1+k(t)} \cdot x(t) \right\} ; \tag{46}$$

with the following optimality conditions:

$$\frac{\partial H(t)}{\partial k(t)} = -\rho - \frac{\lambda(t)}{[1+k(t)]^2} \cdot x(t) = 0 ; \tag{47}$$

$$-\frac{\partial H(t)}{\partial x(t)} = \frac{\partial \beta(t)}{\partial t} \Rightarrow \frac{\partial \lambda(t)}{\partial t} = \left[\rho + \frac{k(t)}{1+k(t)} \right] \lambda(t) - 1 + 2x(t) ; \tag{48}$$

$$\lim_{t \to \infty} \beta(t) \cdot x(t) = 0 : \tag{49}$$

From (47), I can establish that

$$\lambda(t) = -\frac{\rho [1+k(t)]^2}{x(t)} ; \tag{50}$$

$$k(t) = -1 + \frac{1}{\sqrt{\rho}} \sqrt{-\lambda(t) x(t)} ; \tag{51}$$

which entails:

$$\frac{\partial k(t)}{\partial t} \propto -\frac{\partial \lambda(t)}{\partial t} x(t) - \frac{\partial x(t)}{\partial t} \lambda(t) : \tag{52}$$

Using (48) and (50), (52) rewrites as:

$$\frac{\partial k(t)}{\partial t} \propto \rho^2 [1+k(t)]^2 - x(t) [2x(t) - 1] : \tag{53}$$

The roots of the r.h.s. expression in (53) are:

$$k(t) = -1 \pm \frac{\sqrt{x(t)\,[2x(t)-1]}}{\rho}. \tag{54}$$

Obviously I can disregard the negative solution, the steady state investment being:

$$k^{ss}_{SP} = -1 + \frac{\sqrt{x(t)\,[2x(t)-1]}}{\rho} \begin{matrix} > \\ = \\ < \end{matrix} 0 \text{ for } x(t) \begin{matrix} > \\ = \\ < \end{matrix} \frac{1+\sqrt{8\rho+1}}{4}. \tag{55}$$

Notice that

$$\frac{1+\sqrt{8\rho+1}}{4} \in \left(\frac{1}{2};1\right) \quad \text{for all } \rho \in [0;1)\,. \tag{56}$$

This entails that, if $\rho \geq 1$; the R&D investment in product innovation is socially too costly for the planner to undertake it at all. Indeed, at $x(0) = 1$; we have $k^{ss}_{SP} = -1 + 1/\rho < 0$ for all $\rho \geq 1$.

The phase diagram is qualitatively the same as in figure 1, and therefore it is omitted. The above discussion leads to the following:[9]

Proposition 3 Under full market coverage, the social planner reaches a steady state at

$$x^{ss}_{SP} = \frac{1+\sqrt{8\rho+1}}{4} \in \left(\frac{1}{2};1\right) \quad \text{for all } \rho \in [0;1)\,.$$

For all $\rho \geq 1$; the planner does not invest and remains at $x(0) = 1$ forever.

[9] The stability analysis is omitted here as well as in the next section. In both cases, equilibria are saddles.

5.2 Optimal R&D in a profit-seeking monopoly

The monopolists maximises net discounted profits, under the dynamic constraint (10). Accordingly, the monopolist's Hamiltonian is:

$$H(t) = e^{-\rho t} \cdot \left\{ s - [x(t)]^2 - \rho k(t) - \lambda(t) \frac{k(t)}{1 + k(t)} \cdot x(t) \right\} ; \tag{57}$$

with the following optimality conditions:

$$\frac{\partial H(t)}{\partial k(t)} = -\rho - \frac{\lambda(t)}{[1 + k(t)]^2} \cdot x(t) ; \tag{58}$$

$$-\frac{\partial H(t)}{\partial x(t)} = \frac{\partial \beta(t)}{\partial t} \Rightarrow \frac{\partial \lambda(t)}{\partial t} = \left[\rho + \frac{k(t)}{1 + k(t)} \right] \lambda(t) + 2x(t) ; \tag{59}$$

$$\lim_{t \to \infty} \beta(t) \cdot x(t) = 0 . \tag{60}$$

As the monopolist's problem largely replicates the planner's, I can quickly outline the solution. The optimal value of $\lambda(t)$ is defined as in (50), while the state variable evolves according to the same dynamics as in (52), which in this case can be written as follows:

$$\frac{\partial k(t)}{\partial t} \propto \rho^2 [1 + k(t)]^2 - 2 [x(t)]^2 . \tag{61}$$

The roots of the r.h.s. expression in (61) are:

$$k(t) = -1 \pm \frac{x(t)\sqrt{2}}{\rho} . \tag{62}$$

As in the planner's case, the negative solution can be disregarded, the steady state investment being:

$$k_M^{ss} = -1 + \frac{x(t)\sqrt{2}}{\rho} \begin{array}{c} > \\ = \\ < \end{array} 0 \text{ for } x(t) \begin{array}{c} > \\ = \\ < \end{array} \frac{\rho\sqrt{2}}{2} . \tag{63}$$

Notice that
$$\frac{\rho\sqrt{2}}{2} \in \left[\frac{1}{2}; 1\right) \quad \text{for all } \rho \in \left[\frac{\sqrt{2}}{2}; \sqrt{2}\right) : \tag{64}$$

This entails that, if $\rho \geq \sqrt{2}$; the monopolist finds it too costly to invest in product development, with $k^{ss}_M = -1 + \frac{\sqrt{2}}{\rho}$ at $x(0) = 1$: For all $\rho \in \left[0; \frac{\sqrt{2}}{2}\right)$; the monopolist reaches $x^{ss}_M = \frac{1}{2}$ at $k^{ss}_M = -1 + \frac{\sqrt{2}}{2\rho} > 0$:

The above discussion is summarised in the following:

Proposition 4 Under full market coverage, the profit-seeking monopolist reaches a steady state at

$$x^{ss}_M = \frac{\rho\sqrt{2}}{2} \in \left[\frac{1}{2}; 1\right) \quad \text{for all } \rho \in \left[\frac{\sqrt{2}}{2}; \sqrt{2}\right) ;$$
$$x^{ss}_M = \frac{1}{2} \text{ for all } \rho \in \left[0; \frac{\sqrt{2}}{2}\right) :$$

For all $\rho \geq \sqrt{2}$; the planner does not invest and remains at $x(0) = 1$ forever.

6 Planning vs monopoly under full market coverage

First of all, compare k^{ss}_{SP} against k^{ss}_M at $x(0) = 1$: This immediately produces the following result:

Lemma 3 At $t = 0$; the optimal instantaneous investment is larger under monopoly than under social planning, for all admissible values of $\{s; \rho\}$:

Then, by virtue of Propositions 3 and 4, I can establish the following taxonomy:

I. For all $\rho \geq \sqrt{2}$, neither the planner nor the monopolist invest in product innovation, and $x_{SP}(t) = x_M(t) = 1$ forever.

II. For all $\rho \in \left[1, \sqrt{2}\right)$, the monopolist reaches a steady state at $x_M^{ss} = \frac{\rho\sqrt{2}}{2} \in \left[\frac{1}{2}, 1\right)$ while the planner remains at $x(0) = 1$.

III. For all $\rho \in \left(\frac{\sqrt{2}}{2}, 1\right)$, both the planner and the monopolist reach a steady state at $x_i^{ss} \in \left[\frac{1}{2}, 1\right)$, $i = M, SP$, with $x_M^{ss} < x_{SP}^{ss}$.

IV. For all $\rho \in \left(0, \frac{\sqrt{2}}{2}\right)$, $x_M^{ss} = x_{SP}^{ss} = \frac{1}{2}$, where x_M^{ss} is a corner solution with $k_M^{ss} > 0$.

The above taxonomy entails that, for all $\rho \geq 0$ and all $x(t) \in \left[\frac{1}{2}, 1\right]$, $k_M^{ss} > k_{SP}^{ss}$. This leads to the final proposition:

Proposition 5 *Under full market coverage, the monopolist's incentive to carry out product development is always larger than the planner's incentive.*

The above result can be given an intuitive explanation on the following grounds. Under full coverage, the monopolist gets always closer to the average (and median) consumer than the planner does, because this enhances the monopolist's ability to extract surplus from any consumers. On the contrary, in choosing the socially optimal investment path, the planner must tradeoff the reduction in transportation costs against the R&D costs, with total demand being fixed at one anyway. Indeed, the planner locates at 1/2 if and only if $\rho = 0$; that is, discounting and the rental price of capital are both nil.

7 Concluding remarks

I have investigated the optimal R&D investment in product innovation of a single-product …rm operating in a spatial market with a uniform consumer distribution, comparing the steady state behaviour of a pro…t-seeking monopolist versus that of a benevolent social planner. This has been done under the alternative assumptions of partial and full market coverage.

The main results emerging in the partial coverage setting can be summarised as follows. Neither the planner nor the pro…t-seeking monopolist does necessarily aim at developing the product variety preferred by the average (and median) consumer located at 1/2. In general, this obtains only when consumer's gross surplus is su¢ciently large. Otherwise, the steady state product design is the outcome of the tradeo¤ between the incentive to locate as close as possible to the middle of the preference space, and the incentive to save upon R&D costs.

This implies that the established wisdom associated with static models (Spence, 1975; Bonanno, 1987), according to which a planner takes into account the average consumer, does not hold true in a dynamic setup. Indeed, the planner may not produce the variety preferred by the average consumer, in situations where the R&D investment needed to reach 1/2 is socially too costly. Accordingly, in those situations where the monopolist's variety is closer to the average consumer taste than the planner's, the extent of market coverage is larger under monopoly than under social planning. Therefore, welfare maximisation is reached by the planner through a reduction in the R&D expenditure as compared to monopoly.

The above results are reinforced under full market coverage. If all consumers buy, the planner's incentive to innovate is always weaker than the monopolist's, and the planner produces the average (and median) consumer's preferred variety if and only if the rental price of capital and the discount rate are both nil.

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